Guía 3 - Solucionario

Tema: Integrales Indefinidas (Ejercicios Adicionales)
En los siguientes ejercicios calcule la integral indefinida por cualquier método de los vistos en clase:

1. $\int xe^x dx$

Sol:
Haciendo $[u = x, dv = e^x dx] \Rightarrow [du = dx, v = e^x]$.
Entonces,

$$\begin{aligned}\int xe^x dx &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \quad \square\end{aligned}$$

2. $\int (\arctan x) dx$

Sol:
Haciendo $[u = \arctan x, dv = dx] \Rightarrow [du = \frac{1}{1+x^2}dx, v = x]$.
Entonces,

$$\begin{aligned}\int (\arctan x) dx &= uv - \int vdu \\ &= x\arctan x - \int \frac{x}{1+x^2}dx \\ &= x\arctan x - \frac{1}{2}\int \frac{da}{a} \quad [a = 1 + x^2 \Rightarrow da = 2x] \\ &= x\arctan x - \frac{1}{2}\ln a + C \\ &= x\arctan x - \frac{1}{2}\ln(1 + x^2) + C \quad \square\end{aligned}$$

3. $\int \ln 3x \; dx$

Sol: Por partes o inmediatamente se llega a:

$$\int \ln 3x dx = xln(3x) - \frac{x}{3} + C \quad \square$$

4. $\int z^3 \ln z \; dz$

<u>*Sol:*</u>

Haciendo $[u = z^3, dv = \ln(z)dz] \Rightarrow [du = 3z^2dz, v = z\ln(z) - z]$.

Entonces,

$$\begin{aligned}\int z^3 \ln z dz &= uv - \int vdu \\ &= z^3(z\ln z - z) - 3\int (z\ln z - z)z^2 dz \\ &= z^3(z\ln z - z) - 3\int z^3 \ln z dz + 3\int z^3 dz \\ &= z^3(z\ln z - z) - 3\int z^3 \ln z dz + 3\left(\frac{z^4}{4}\right) + C\end{aligned}$$

, luego:

$\int z^3 \ln z dz = \dfrac{z^3(z\ln z - z)}{4} + \dfrac{3z^4}{16} + C = z^4\left(\dfrac{\ln z - 1}{4} + \dfrac{3}{16}\right) + C$

$= z^4\left(\dfrac{4\ln z - 1}{16}\right) + C \quad \square$

5. $\int w \ln w \; dw$

<u>*Sol:*</u>

Haciendo $[u = w, dv = \ln(w)dw] \Rightarrow [du = dw, v = w\ln w - w]$.

Entonces,

$$\begin{aligned}\int w\ln w dw &= uv - \int vdu \\ &= w(w\ln w - w) - \int (w\ln w - w)dw \\ &= w^2\ln w - w^2 - \int w\ln(w)dw + \int wdw \\ \Rightarrow 2\int w\ln w dw &= w^2\ln w - w^2 + \frac{w^2}{2} + C \\ \Rightarrow \int w\ln w dw &= \frac{w^2}{4}(2\ln w - 1) + C \quad \square\end{aligned}$$

6. $\displaystyle\int \csc^3 x\ dx$

<u>*Sol:*</u>

$$\begin{aligned}
\int \csc^3 x dx &= \int \csc x \csc^2 x dx \\
&= \int \csc x(1+\cot^2 x)dx \\
&= \int \csc x dx + \int \csc x \cot x \cot x dx \\
&= \int \csc x dx - \cot x \csc x - \int \csc^3 x dx \\
&[u=\cot x, dv = \csc x \cot x dx] \Rightarrow [du = -\csc^2 x dx, v = -\csc x]
\end{aligned}$$

, luego:

$\displaystyle\int \csc^3 x dx = -\frac{\cot x \csc x + \ln(\cot x + \csc x)}{2} + C \quad \square$

7. $\displaystyle\int x \sin^2 x\ dx$

<u>*Sol:*</u>

Por partes $\left[u = x, dv = \sin^2 x dx = \dfrac{1-\cos(2x)}{2}dx\right] \Rightarrow \left[du = dx, v = \dfrac{2x - \sin(2x)}{4}\right]$.

Entonces,

$$\begin{aligned}
\int x\sin^2 x dx &= uv - \int v du \\
&= x\left(\frac{2x-\sin(2x)}{4}\right) - \int \frac{2x-\sin(2x)}{4}dx \\
&= x\left(\frac{2x-\sin(2x)}{4}\right) - \frac{x^2}{4} - \frac{\cos(2x)}{8} + C \\
&= \frac{2x^2 - 2x\sin(2x) - \cos(2x)}{8} + C \quad \square
\end{aligned}$$

8. $\int x^2 e^x \, dx$

<u>*Sol:*</u>

Por partes $[u = x^2, dv = e^x dx] \Rightarrow [du = 2xdx, v = e^x]$.

Entonces,

$$
\begin{aligned}
\int x^2 e^x dx &= uv - \int v du \\
&= x^2 e^x - 2 \int x e^x dx \\
&= x^2 e^x - 2(x e^x - \int e^x dx) \quad [u = x, dv = e^x dx \Rightarrow du = dx, v = e^x] \\
&= x^2 e^x - 2(x e^x - e^x) + C \\
&= e^x (x^2 - 2x + 2) + C \quad \square
\end{aligned}
$$

9. $\int \ln^2 x \, dx$

<u>*Sol:*</u>

Por sustitución $[u = \ln x] \Rightarrow [x = e^u, dx = e^u du]$.

Entonces,

$$
\begin{aligned}
\int \ln^2 x dx &= \int u^2 e^u du \\
&= e^u (u^2 - 2u + 2) + C \quad \text{[Ejercicio 8]} \\
&= x(\ln^2 x - 2 \ln x + 2) + C \quad \square
\end{aligned}
$$

10. $\int e^t \cos t \, dt$

<u>*Sol:*</u>

Por partes $[u = \cos t, dv = e^t dt] \Rightarrow [du = -\sin t dt, v = e^t]$.

Entonces,

$$\begin{aligned}\int e^t \cos t dt &= uv - \int v du \\ &= e^t \cos t + \int e^t \sin t dt \\ &= e^t \cos t + e^t \sin t - \int e^t \cos t dt \quad [u = \sin t, dv = e^t dt \Rightarrow du = \cos t dt, v = e^t\end{aligned}$$

$$\Rightarrow \int e^t \cos t dt = \frac{e^t \cos t + e^t \sin t}{2} + C \quad \square$$

11. $\displaystyle\int (\ln x)^3 \, dx$

<u>*Sol:*</u>

Primero se hace la sustitución $[y = \ln x] \Rightarrow [x = e^y, dx = e^y dy]$.

Entonces,

$$\begin{aligned}\int \ln^3 x dx &= \int y^3 e^y dy \\ &= y^3 e^y - 3\int y^2 e^y dy \quad [\text{Partes:} u = y^3, dv = e^y dy \Rightarrow du = 3y^2, v = e^y] \\ &= y^3 e^y - 3e^y(y^2 - 2y + 2) + C \quad [\text{Ejercicio } 8] \\ &= x \ln^3 x - 3x(\ln^2 x - 2\ln x + 2) + C \quad \square\end{aligned}$$

12. $\displaystyle\int \sqrt{\tan x} \, dy$

<u>*Sol:*</u>

La integral es con respecto a y, entonces

$$\int \sqrt{\tan x} \, dy = \sqrt{\tan x} \int dy = \sqrt{\tan x} \, [y + C] \quad \square$$

13. $\displaystyle\int x\sqrt{x+3} \, dx$

<u>*Sol:*</u>

Por partes $u = x, dv = \sqrt{x+3} \quad dx \Rightarrow \left[du = dx, v = \frac{2}{3}(x+3)^{3/2}\right]$.

Entonces,

$$\begin{aligned}\int x\sqrt{x+3}dx &= uv - \int vdu \\ &= x\left(\frac{2}{3}(x+3)^{3/2}\right) - \int \frac{2}{3}(x+3)^{3/2}dx \\ &= x\left(\frac{2}{3}(x+3)^{3/2}\right) - \frac{4}{15}(x+3)^{5/2} + C \\ &= \frac{2}{3}(x+3)^{3/2}\left(x - \frac{2}{5}(x+3)\right) + C \quad \square\end{aligned}$$

14. $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$

Sol:

Por sustitución trigonométrica $[x = \sin\theta] \Rightarrow [dx = \cos\theta d\theta]$.

Entonces,

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x}dx &= \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta d\theta \\ &= \int \frac{\cos\theta}{\sin\theta}\cos\theta d\theta \\ &= \int \frac{\cos^2\theta}{\sin\theta}d\theta \\ &= \int \frac{1-\sin^2\theta}{\sin\theta}d\theta \\ &= \int (\csc\theta - \sin\theta)d\theta \\ &= -\ln|\csc\theta + \cot\theta| + \cos\theta + C \\ &= -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C \quad \text{[Hacer triángulo]} \quad \square\end{aligned}$$

15. $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$

Sol:

Por sustitución trigonométrica $[x = 3\tan\theta] \Rightarrow [dx = 3\sec^2\theta d\theta]$.

Entonces,

$$\begin{aligned}\int \frac{dx}{(x^2+9)^{3/2}} &= \int \frac{3\sec^2\theta d\theta}{(9\tan^2\theta+9)^{3/2}}\\ &= \int \frac{3\sec^2\theta d\theta}{(9\sec^2\theta)^{3/2}}\\ &= \frac{1}{9}\int \frac{\sec^2\theta d\theta}{\sec^3\theta}\\ &= \frac{1}{9}\int \cos\theta d\theta\\ &= \frac{1}{9}\sin\theta + C\\ &= \frac{x}{9\sqrt{x^2+9}} + C \quad \text{[Hacer triángulo]} \quad \square\end{aligned}$$

16. $\int \frac{dx}{x^2\sqrt{x^2-16}}$

Sol:

Por sustitución trigonométrica $[x = 4\sec\theta] \Rightarrow [dx = 4\sec\theta\tan\theta d\theta]$.

Entonces,

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-16}} &= \int \frac{4\sec\theta\tan\theta d\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\\ &= \int \frac{\tan\theta d\theta}{16\sec\theta|\tan\theta|}\\ &= \int \frac{d\theta}{16\sec\theta}\\ &= \int \frac{\cos\theta d\theta}{16}\\ &= \frac{\sin\theta}{16} + C\\ &= \frac{\sqrt{x^2-16}}{16x} + C \quad \square\end{aligned}$$

17. $\int \frac{\sqrt{t^2-4}}{t^3}dt$

Sol:

Por sustitución trigonométrica $[t = 2\sec\theta] \Rightarrow [dt = 2\sec\theta\tan\theta d\theta]$.

Entonces,

$$\begin{aligned}\int \frac{\sqrt{t^2-4}}{t^3}dt &= \int \frac{\sqrt{4\sec^2\theta-4}}{8\sec^3\theta}2\sec\theta\tan\theta d\theta = \frac{1}{2}\int \frac{|\tan\theta|}{\sec^2\theta}\tan\theta d\theta \\ &= \frac{1}{2}\int \frac{\tan^2\theta}{\sec^2\theta}d\theta \\ &= \frac{1}{2}\int \frac{\sec^2\theta-1}{\sec^2\theta}d\theta = \frac{1}{2}\int(1-\cos^2\theta)d\theta = \frac{1}{8}\int(2-2\cos 2\theta)d\theta \\ &= \frac{1}{8}(2\theta-\sin 2\theta)+C = \frac{1}{4}\left(\sec^{-1}(t/2)-\frac{2\sqrt{t^2-4}}{t^2}\right)+C \quad \text{[Hacer triángulo]}\end{aligned}$$

Luego,

$$\begin{aligned}\int \frac{\sqrt{t^2-4}}{t^3}dt &= \frac{1}{4}\left(\sec^{-1}(t/2)-\frac{2\sqrt{t^2-4}}{t^2}\right)\Bigg|_2^5 \\ &= \frac{1}{4}\left(\sec^{-1}(5/2)-\frac{2\sqrt{21}}{25}\right) \quad \square\end{aligned}$$

18. $\displaystyle\int \frac{2x-1}{x^2-6x+18}\,dx$

Sol:

$$\begin{aligned}\int \frac{2x-1}{x^2-6x+18}dx &= \int\left(\frac{2x-6}{x^2-6x+18}+\frac{5}{x^2-6x+18}\right)dx \\ &= \ln|x^2-6x+18| + \frac{5}{9}\int \frac{dx}{(\frac{x-3}{3})^2+1}+C \\ &= \ln|x^2-6x+18| + \frac{5}{9}\int \frac{3\sec^2 t}{\tan^2 t+1}dt + C \\ &\left[\frac{x-3}{3}=\tan t \Rightarrow dx = 3\sec^2 t dt\right] \\ &= \ln|x^2-6x+18| + \frac{5}{3}\int \frac{\sec^2 t}{\sec^2 t}dt + C \\ &= \ln|x^2-6x+18| + \frac{5}{3}t + C \\ &= \ln|x^2-6x+18| + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right)+C\end{aligned}$$

$$\Rightarrow \int \frac{2x-1}{x^2-6x+18}dx = \left[\ln|x^2-6x+18| + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right)\right]_{\sqrt{3}}^{\sqrt{6}}$$

$$= \ln\left(\frac{24-6\sqrt{6}}{21-6\sqrt{3}}\right) + \frac{5}{3}\left[\tan^{-1}\left(\frac{\sqrt{6}-3}{3}\right) - \tan^{-1}\left(\frac{\sqrt{3}-3}{3}\right)\right] \quad \square$$